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DYNAMIC PROBLEMS FOR THE SINE-GORDON EQUATION WITH VARIABLE COEFFICIENTS. EXACT SOLUTIONS†

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The sine-Gordon equation with dissipation and a variable coefficient on the non-linear term is considered. This equation describes waves in an energetically open system with an external field acting on it which varies monotonically with time. Scale transformations, matched with the external field and the dissipation, are introduced which reduce the generalized equation to the standard equation. It is shown that processes for controlling the oscillations and waves exist for which the equation is transformed to a form with constant coefficients and an effective dissipation which vanishes or is either positive (damping of the oscillations) or negative (their amplification). Waves, which propagate with a constant, decaying or increasing amplitude and variable frequency and velocity correspond to them.

1. BASIC RELATIONS

The non-linear dynamics of certain extended elastic systems with dissipation is described by the sine-Gordon equation (below, both signs are meaningful)

$$j\Phi_{tt} = k\Delta\Phi \pm f^2(t)\sin\Phi - \gamma\Phi_t \tag{1.1}$$

The subscripts denote derivatives with respect to the time t and Δ is the Laplace operator. The positive constant coefficients j, k, γ characterize the inertia, elasticity and dissipation respectively. The coefficient $f^2(t)$ represents the elastic energy of a homogeneous configuration or an external force. The required function $\Phi(t, x, y, z)$ has the meaning of a displacement field (angular or linear) which is unimportant for the subsequent formal analysis.

A model of an ensemble of elastically connected pendulums or oscillators formally corresponds to this equation. Real examples are liquid crystals, ferromagnetica materials, crystal lattices and other periodic structures, and, also, mechanical objects such as a flexible rod in a gravitational field, a membrane on a non-linear elastic base, etc. It is important that the topic of discussion is an open system with energy exchange due to dissipation and an external field.

Special processes for controlling oscillations and the motion in general of the system of pendulums in question using an external force $f(t)$ are of interest. Vibration control processes are known which lead to compensation of the damping and amplification of the oscillations as a result of resonance or their damping (in the case of antiresonance). It is clear that, here, we are dealing with a parametric equation. We are interested in the effects of the amplification or damping of oscillations but in a monotonic control process.

As it applies to Eq. (1.1), an approach, within the limits of which scale transformations are introduced which are matched with the external field and the dissipation, is effective.

In certain cases, this allows the procedure for constructing the solution of Eq. (1.1) to be simplified considerable, and enables interesting systems of forced oscillations to be revealed.

We will write Eq. (1.1) in the reduced coordinates x, χ, η, ξ which are related to t, x, y, z as follows:

$$s = \frac{1}{\sqrt{j}}\int f(t)dt, \quad \chi = \frac{xf(t)}{\sqrt{k}}, \quad \eta = \frac{yf(t)}{\sqrt{k}}, \quad \xi = \frac{zf(t)}{\sqrt{k}} \tag{1.2}$$

Self-adjusting scales for all of the coordinates are thereby introduced which take account of, in particular, the characteristic "time" s as a function of γ and $f(t)$. We have the obvious relations

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$$j\Phi_{tt} = f^2\Phi_{ss} + \sqrt{j}f_t\Phi_s,\ \ \Phi_t = f\Phi_s/\sqrt{j},\ \ k\Delta = f^2\bar{\Delta} \tag{1.3}$$

The subscript s denotes a derivative with respect to the corresponding coordinate and $\bar{\Delta}$ is the Laplace operator in the reduced coordinates ξ, χ, η. Equation (1.1) then takes the form

$$\Phi_{ss} = \bar{\Delta}\Phi \pm \sin\Phi - G(t)\Phi_s \tag{1.4}$$

Here

$$G(t) = \frac{1}{\tau\omega_f}\left[1 + \frac{\tau}{2}(\ln f^2)_t\right],\ \ \tau = \frac{j}{\gamma},\ \ \omega_f = \frac{f}{\sqrt{j}} \tag{1.5}$$

We shall regard Eq. (1.4) as the sine-Gordon equation with an effective dissipation ($G > 0$) or with energy capture and pumping ($G > 0$). Both versions are realizable depending on the rate of change of the external force.

In order to refine the meaning of the quantity $G(t)$, we turn to the first equality in (1.3). On multiplying it by Φ_t, after some elementary reduction we obtain

$$2Q_t = f^3\sqrt{j}(\Phi_s^2\ )_s + (f^2)_t\Phi_s^2\sqrt{j},\ \ 2Q = j\Phi_t{}^2 \tag{1.6}$$

Here Q is the kinetic energy of the oscillations and the second term on the right-hand side is the addition to its rate of change due to a variable external force. It can be seen that, on the basis of relations (1.5) and (1.3),

$$\Phi_t{}^2G = \frac{2\gamma\Phi_t{}^2 + (f^2)_t\Phi_s{}^2}{2f\sqrt{j}} \tag{1.7}$$

Hence, the quantity $G(t)$ in the reduced equation (1.4) is the energy exchange coefficient for an open elastoviscous system with an external field and a thermal "reservoir". It is important that unlike dissipation, energy exchange with a field is not of fixed sign. If the rate (f^2) is positive, energy leaves the system and, as in the case of dissipation, increases the damping of the motion. Otherwise, energy capture, that is, the pumping of energy, occurs which may compensate for damping of a dissipative nature.

2. COMPLETE COMPENSATION OF DISSIPATIVE FORCES

Complete compensation is achieved when the quantity G vanishes, that is, when the condition

$$-(\ln f^2)_t = 2/\tau,\ \ G = 0 \tag{2.1}$$

is satisfied.

In this case, the negative rate of change of the external field is equal to twice the relaxation frequency $1/\tau$ of the free oscillations. A similar condition holds in the case of parametric resonance at a real (and not an imaginary) frequency of the external field. In the case in question, there is an unusual kinetic resonance, that is, maximally intense action on the relaxing system.

The undamped nature of the motion (with respect to the amplitude), which is a consequence of the pumping of energy from elsewhere, only occurs, as follows from condition (1.2), for a particular control process when

$$f(t) = f_0\exp(-t/\tau),\ \ f_0 = f(0),\ \ G = 0 \tag{2.2}$$

In this case, Eq. (1.4) takes the form

$$\Phi_{ss} = \bar{\Delta}\Phi \pm \sin\Phi \tag{2.3}$$

It only has constant coefficients and a number of its interesting particular solutions can be obtained. For instance, the exact solution in the bounded domain $10 \leqslant \chi \leqslant \chi_0$ for null boundary conditions has been found in [1]. It has the form

$$\operatorname{tg}\frac{\Phi}{2} = a\,\mathrm{cn}\frac{K(\nu_1)\chi}{\chi_0}\mathrm{cn}(sK(\nu_2)),\ \ s(0)=0 \tag{2.4}$$

Here, on denotes an elliptic cosine function, and $K(\nu_1)$ and $K(\nu_2)$, which are complete elliptic integrals of the first kind, are functions of the moduli $0 \leqslant \nu_1, \nu_2 < 1$.

The periodic orientational oscillations of the structure of a nematic liquid crystal in a layer of thickness h with null boundary conditions in a constant magnetic field $H = f_0$ have been considered in [1]. The amplitude a is constant and the second elliptic cosine function oscillates in time, the role of which is played by the variable s [1]. In the case of a variable field and dissipation, the dependence of this variable on the absolute time t introduces the effect of an external force. In accordance with relations (1.2) and (2.2), we can write

$$\frac{\chi}{\chi_0} = \frac{x}{h},\ \ s = \tau\omega_0\left[1-\exp\left(-\frac{t}{\tau}\right)\right] = \tau\omega_0\left[\frac{t}{\tau}+O\left(\frac{t^2}{\tau^2}\right)\right] \tag{2.5}$$

Here, the exponent is represented by a power series and $O(t^2/\tau^2)$ is its residue; when $t \ll \tau$, it can be neglected and solution (2.4) represents forced standing oscillations, which are periodic in space and time, as they also were in the case considered earlier in [1]. Otherwise, periodic oscillations with a phase modulation are obtained which are also the subject of analysis.

Having a variable field $f(t)$ in mind, we will consider all the terms of the series in expansion (2.5). Taking account of the asynchronous nature of the oscillations, we introduce the periods of motion T_n. We define them as the successive "distances" between the zeros of the elliptic cosine by putting $s = n$ ($n = 1, 3, 5, \ldots$). Then

$$\tau\omega_0\left[1-\exp\left(-\frac{t}{\tau}\right)\right] = n,\ \ -\frac{T_n}{\tau} = \ln\left((1-\frac{n}{\tau\omega_0}\right) \tag{2.6}$$

It is obvious that the periods increase as the number of oscillations in increases, and the more strongly the shorter the characteristic time τ and the smaller the dissipation coefficient γ. An increase in the periods can lead to a change in the nature of the motion. So long as $T_n \ll \tau$, it has a periodic form but it becomes a periodic after the threshold $T_n \approx \tau$ has been surmounted. The amplitude factor $a\ \mathrm{cn}(xK_1/h)$ remains unchanged.

The forced oscillations (by the external field) differ from the free oscillations in that, when the field f is constant, dissipation leads in the first place to damping the amplitude. The increase in their periods is a secondary effect.

3. AMPLIFICATION AND DAMPING MODES

We will now consider other control processes in which damping of the amplitude of the oscillations occurs that differs from the dissipative process or, conversely amplification of these oscillations is induced. Formally, they follow from the condition $G(t) = G_0$, where the constant can be both positive (damping of the oscillations) and negative (their amplification). As a result, the equation of the oscillations again takes the form of an equation with constant coefficients

$$\Phi_{ss} = \overline{\Delta}\Phi \pm \sin\Phi - G_0\Phi_s \tag{3.1}$$

It describes both decaying as well as increasing oscillations depending on the sign of the coefficient G_0, which has already been mentioned above. Its solutions for an infinite, one-dimensional space are constructed, as is well known, using asymptotic methods [2].

The control function $f(t)$ is found from the equation obtained from relations (1.5) by replacing $G(t)$ by G_0, namely

$$(\ln f^2)_t = \frac{2G_0 f}{\sqrt{j}} - \frac{2}{\tau} \tag{3.2}$$

Its solution has the form

$$f(t)=\frac{\sqrt{j}}{G_0\tau}\left[1-\left(1-\frac{\sqrt{j}}{f_0G_0\tau}\right)\exp\left(\frac{t}{\tau}\right)\right]^{-1} \tag{3.3}$$

The form of the corresponding curves $f(t)$ is defined by the ratio of the parameters G_0, τ and f_0. We will first consider the case of the amplification of the oscillations $G_0 < 0$. Solution (3.3) then gives a sharply decaying monotonic curve $f(t)$ which is such that

$$f(t)\to 0,\ t\to\infty,\ f(t)\to\infty,\ t\to t_c,\ t_c=-\tau\ln\left(1-\frac{1}{f_0G_0\tau}\right)<0 \tag{3.4}$$

Again, as in the case of complete compensation, $f(0)=f_0$ and the frequency of the forced oscillations falls to zero with time. However, time values $t < t_c$ are now excluded. Finally, in order to reveal the growth in the amplitude, it is necessary to solve Eq. (3.1) taking account of the fact that $G_0 < 0$.

We will consider the case of the damping of the oscillations $G_0 > 0$, which is represented by two modes, soft and hard. In the first case, $G_0 < \sqrt{j}/(f_0\tau)$. The function $f(t)$, which is defined by expression (3.3), is then a monotonically decreasing bounded function, where

$$f\to 0,\ t\to\infty,\ f\to\frac{\sqrt{j}}{G_0\tau},\ t\to-\infty,\ f(0)=f_0 \tag{3.5}$$

It is also obvious that the frequency of the forced oscillations changes within fixed limits, decaying from the finite value of $1/(G_0\tau)$ to zero in the same way as the amplitude.

The hard damping mode arises when $G_0 > \tau\sqrt{j}/f_0$. Then, $f(t)$ is a function which increases monotonically from the value $f=\sqrt{j}/(G_0\tau)$ when $t\to-\infty$ to infinity when $t\to t_c$, where t_c is calculated using formula (3.4), but taking into account that G_0 is positive. An unbounded increase in the frequency of the forced vibrations from the finite value $1/(G_0\tau)$ also corresponds to this growth and, moreover, they decay in amplitude. The latter fact is also revealed on solving Eq. (3.1), where $G_0 > 0$. These conclusions were reached earlier for small values of G_0 in [2].

4. CONTROL OF A WAVE IN THE COMPENSATION MODE

We will consider the control of the propagation of a two-dimensional wave $\Phi(x,y,t)$ in the compensation mode. Using (2.1), we obtain for the reduced variables

$$s=\frac{f_0\tau}{\sqrt{j}}\left[1-\exp\left(-\frac{t}{\tau}\right)\right],\ \chi=\frac{xf_0}{\sqrt{k_1}}\exp\left(-\frac{t}{\tau}\right),\ \eta=\frac{yf_0}{\sqrt{k_2}}\exp\left(-\frac{t}{\tau}\right),\ \tau=\frac{j}{\gamma} \tag{4.1}$$

We shall seek a solution of Eq. (2.3) by introducing the two independent arguments

$$\chi,\ \Theta=\eta+vs,\quad v<1 \tag{4.2}$$

Here, v is the velocity of the (slow) (η, s)-wave. Then, Eq. (2.3) takes the form of the sine–Helmholtz equation

$$(1-v^2)\frac{\partial^2\Phi}{\partial\Theta^2}+\frac{\partial^2\Phi}{\partial\chi^2}-\sin\Phi=0,\ v<1 \tag{4.3}$$

The sine-Gordon equation would be obtained in the case of a fast wave ($v > 1$).

In any case, the variable phase of the wave in the space χ, η, s is connected with the initial (real) variables y and t as follows:

$$l_2\Theta(y,t)=(y+vC_2\tau)\exp\left(-\frac{t}{\tau}\right)-vC_2\tau,\ l_2=\sqrt{\frac{k_2}{f_0}},\ C_2=\sqrt{\frac{k_2}{j}},\ \tau=\frac{j}{\gamma} \tag{4.4}$$

It is important that there is a periodic field (with respect to y) at the initial instant of time $t=0$ since $\Theta(y,0)=y/l_2$. Suppose its spatial period is equal to λ. At successive instants of time when $t \ll \tau$, we have, on expanding the exponent in series

$$l_2\Theta(y,t) = y - vC_2(t+\tau) + yO\left(\frac{t^2}{\tau^2}\right) + \frac{yt}{\tau} \tag{4.5}$$

The first two terms, which are linear in y and t, determine the periodic component of the wave, while the second two components determine its modulation.

In order to refine the properties of the wave motion, we introduce its phase velocity C_0 as the velocity of motion of a certain constant value of the phase Θ_c. It is obvious that the law of motion, according to relation (4.4), has the form

$$y + v\,C_2\tau = (l_2\Theta_c + v\,C_2\tau)\exp\frac{t}{\tau},\quad \Theta_c = \Theta(y,0) = \frac{y(0)}{l_2} \tag{4.6}$$

We then obtain the exponentially increasing phase velocity

$$C_0 = \frac{\partial y}{\partial t} = \left(\frac{l_2\theta_c}{\tau} + v\,C_2\right)\exp\frac{t}{\tau} = \frac{y}{\tau} + v\,C_2 \tag{4.7}$$

It is clear that Θ_c is the phase value at an arbitrary point y at the initial instant of time. At the point $y = y_0 = -vC_2\tau$, we have $C_0 = 0$, that is, the phase value of this point, which is equal to $\Theta_0 = -\,vC_2\tau$, remains constant in time. This can also be seen directly from relation (4.4). The quantity

$$\delta\Theta = \Theta_c + v\,C_2\tau / l_2 \tag{4.8}$$

is the difference in the phases at the points $y(0)$, y_0, that is, the difference between the phases of the oscillations of the corresponding pendulums at the initial instant of time.

It is clear that, with time, this difference in the phases departs to infinity (to the left or right of y_0) such that $C_0 > 0, y > y_0$ and $C_0 < 0, y < y_0$, and becomes equal to zero there. This process, as can be seen from relation (4.4), is concluded when $t \to \infty$ by the fact the phase value $\Theta_0 = -\,vC_2\tau$ fills the whole space, that is, $\Theta(y, t) \to \Theta_0, t \to \infty$.

An equalization of the wave field occurs, that is, there is a decrease in the gradients and an elongation of the wave. Actually, on turning to relations (4.6), we find the law for the growth of the wavelength with time. Defining it as the distance Λ between the closest zeros of the function $\Phi[\Theta(y, 0)]$, we obtain

$$\Lambda = \lambda\exp\frac{t}{\tau} \tag{4.9}$$

Here λ is the period of the function $\Phi(y, 0)$.

It is interesting that the elongation of the spatial period with time and the increase in the phase velocity are compatible with the invariance of the period of the oscillations T. This period is found by integrating the phase velocity (4.7) within the limits from t to $t + T$.

Then

$$\lambda = -\lambda_2\delta\Theta\left[1 - \exp\frac{T}{\tau}\right]\exp\frac{t}{\tau} \tag{4.10}$$

Comparing this expression with the earlier expression, we find, for T, the expression

$$\exp\frac{T}{\tau} = 1 + \frac{\lambda}{l_2\delta\Theta} \tag{4.11}$$

It does not contain variable quantities. However, there is an arbitrary value of the phase difference, which increases linearly along a coordinate at the initial instant of time.

If the period is small, we are dealing with an oscillatory process. Then

$$T/\tau \approx \lambda/(l_2\delta\Theta),\quad T \ll \tau \tag{4.12}$$

It is clear that the phase difference must be sufficiently large, that is

$$\delta\Theta \gg \lambda/l_2 \tag{4.13}$$

Otherwise, the motion has an aperiodic form.

The condition $\delta\Theta \approx \lambda/l_2$ approximately determines the threshold or point on the Oy axis which separates the domains of oscillatory and aperiodic motion. The domain of aperiodic motion is concentrated around the point y_0, at which there is no motion. Its length along the axis is equal to $\delta\Theta l_2 \approx \lambda$. Over this section accelerating travelling (in both directions) coherence waves of the pendulums are generated. When $t \to \infty$, complete in-phase behaviour is established, that is, $\Theta \to -vC_2\tau/l_2$, but the period of the oscillations remains unchanged.

The elongation of the wave with time is associated with a decrease in the external field and an increase in the coherence scale $l \sim \sqrt{k/f}$, as a result of which the non-linear sine-Gordon equation (1.1) degenerates into a linear wave equation. There is no continuous transition of one of the solutions into another. When $l \to \infty$, we approach a bifurcation point, but without passing through it. It separates the branch of long-wave acoustic oscillations and the original short-wave pseudo-optical oscillations, which degenerate as it accelerates into a second acoustic mode as the external field disappears.

Up to now, one sign of the velocity v has been considered. In fact, two (η, s)-wave processes are possible. The second process likewise involves a pair of waves, but they originate from a point which is symmetrical about the origin of the Oy axis. In other words, two pairs of waves, that depart to infinity, emerge from the points $y_0 = \pm vC_2\tau$. As a result, in the section between these points, there are two waves which travel in opposite directions with opposite phases.

5. AN ACCELERATING STRUCTURAL-REORGANIZATION WAVE

From results previously obtained [3], an exact solution of Eq. (4.3) (with a minus sign in front of the sine) for a structural-reorganization wave of a two-dimensional crystal lattice can be found:

$$\operatorname{tg}\frac{\Phi}{4} = \frac{\operatorname{tn}[xK(\nu_1)h^{-1}]}{A\operatorname{sn}[(\eta+vs)K(\nu_2)\lambda^{1}]} \tag{5.1}$$

We mean by Φ, the relative displacement (along a layer, along the Oy axis) of neighbouring atoms [3], and tn and sn are the Jacobian elliptic tangent and elliptic sine respectively. At the initial instant of time, relation (5.1) reduces to the static solution, obtained for the case [3, 4] when the slippage of neighbouring atomic chains at one or several interatomic distances is specified on the boundaries of each strip of thickness h. The value of Φ is related to the period of the lattice.

Expression (5.1) is an exact solution of Eq. (1.1) on changing to the variables x, y, t. It represents a non-linear wave of the microdisplacements of the atoms (a pseudo-optical mode of structural reorganizations) which propagates along a layer when a shear band (of thickness h) is formed in a crystal. The microdeformation in it is modulated along the Oy axis by the formation of transverse domain walls, composed of incompatibility solutions caused by the breakdown of translational order.

If the interatomic potential barriers do not change with time, then, as can be seen from relations (4.1) and (4.2). $s \sim t$, $\eta \sim y$, and the phase of the wave $(\eta + vC_2t)$ is linear with respect to time and the spatial coordinates. We are then dealing with a periodic non-linear wave in the variables x, y, t which propagates at constant velocity vC_2 and with amplitude $\operatorname{tn}[xK(\nu_1)/h]$ along the layer. This is also a pseudo-optical, but stationary, wave. The velocity, frequency and wavelength, i.e. the distances between transverse defects λ, are conserved.

The interatomic barriers can decrease according to the law $f^2(t)$ both due to external (macroscopic) deformations as well as due to the temperature. Then, a crystal approaches a point of structural transition [4] at which the interlayer shear strength or the structural stability is sharply reduced.

The distinctive features of solution (5.1), above all, the sharp increase in the velocity of the wave and its acceleration are, in fact, associated with this. It appears that dissipation in the lattice, which is taken account of in solution (5.1), also does not prevent this. At the same time, the phase of the wave is found to be a non-linear function of the time in accordance with relation (4.4). There is phase modulation and an increase in the velocity (up to the speed of sound) and wavelength with time as it propagates. The distance between the above-mentioned defects also increases, and, in the final limit, when the interatomic barriers disappear, the defects depart to infinity and the laminar structure of a new phase is formed [4], while the pseudo-optical mode degenerates into a (second) acoustic mode, which accompanies the phase transformation. It is well known that the growth of a martensite phase in certain metals has an explosive nature. However, further analysis is required here.

Hence, the generalized sine-Gordon equation (1.1) has a number of exact solutions that correspond to certain external fields $f(t)$, which increase and decay with time and which create the effects of compensation of mechanical losses in an energetically open system or, conversely, the effects of their activation. The proposed method of adaptive scaling, based on the introduction of reduced variables, enables one to take account of the balance of the inflow (outflow) of energy due to an external field and the dissipation in an open system. As a result, its description is simplified and reduces to the solutions of the classical sine-Gordon equation with constant coefficients. At the same time, the external fields ensure monotonic parametric control of an open system using the principle of the least or greatest mechanical losses.

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